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Invariant resolutions for several Fueter operators

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Abstract

A proper generalization of complex function theory to higher dimension is Clifford analysis and an analogue of holomorphic functions of several complex variables were recently described as the space of solutions of several Dirac equations. The four-dimensional case has special features and is closely connected to functions of quaternionic variables. In this paper we present an approach to the Dolbeault sequence for several quaternionic variables based on symmetries and representation theory. In particular we prove that the resolution of the Cauchy–Fueter system obtained algebraically, via Gröbner bases techniques, is equivalent to the one obtained by R.J. Baston (J. Geom. Phys. 1992).

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1. Introduction

It is now widely accepted that a proper generalization of complex function theory is described by Clifford analysis, which is the function theory for solutions of the Dirac operator. In higher dimension, functions on $\mathbb{R}^n$ with values in the Clifford algebra (resp. in the corresponding spinor space) are considered instead of complex valued functions.

In recent years, a lot of attention was devoted to some problems related to the generalization of the theory of several complex variables to higher dimensions. By that we mean the study of solutions of two or more Dirac operators defined on two or more copies of $\mathbb{R}^n$ with values in the Clifford algebra (resp. in the spinor space).

The Dirac operator on an Euclidean space is elliptic, like the Cauchy–Riemann operator. For this reason in the case of several variables, we expect to encounter phenomena of Hartogs type, as it happens in the complex case. This can be appropriately described by studying an analogue of the Dolbeault sequence. The first map of this sequence, in sufficiently high dimension, is given by several Dirac operators.

A lot of interesting results were already obtained in this direction. Some of the methods used came either from algebraic analysis (Hilbert Syzygy Theorem, Gröbner bases, see [10]) and, in the case of several Dirac operators, from Clifford analysis (megaforms, [17]). In this paper, we would like to suggest another approach, based on symmetry considerations and representation theory.

The main idea can be expressed as follow. In Algebraic Analysis, little or no attention is paid to invariance properties of the involved operators. Algebraic methods offers a general construction of a resolution of a given overdetermined differential operator. In the case of higher dimension, there are formidable computational problems connected with this general construction. We would like to show that if the operator to be resolved has a known symmetry, it is possible to use this information to efficiently reduce the computational complexity of the problem.

In general, if the first operator in the sequence is invariant with respect to a given action, the same can be assumed also for all the operators appearing in the resolution. To apply this idea to the construction of higher dimensional analogues of the Dolbeault sequences, one has to find a symmetry for the operator defined by several Dirac derivatives. We are going to study this question in real dimension four, i.e. in the quaternionic case.

Quaternionic geometry is a special case of the so called *parabolic geometries*. A short review of parabolic geometries will be given in Section 4. An extensive series of complexes composed by invariant differential operators was constructed by Baston in [4]. Such a construction is tightly related to the resolution constructed in [1–3] and the form of the resolution can be efficiently deduced for any number of variables. We will give the proofs of these facts without the use of the heavy machinery and technicalities appearing in [4], and by cutting the number of computation implied in [1–3] by a factor of two.

2. Notations

We denote by $\mathbb{H}$ the algebra of quaternions and by $q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ a quaternion, where $x_\ell \in \mathbb{R}$ for $\ell = 0, \ldots, 3$. We will make the identification $\mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$ so that we can write $q = u_1 + \mathbf{j}u_2$ where $u_1 = x_0 + \mathbf{i}x_1$ and $u_2 = x_2 - \mathbf{i}x_3$. The algebra of quaternions can also be represented by 2×2 matrices with complex entries. For $A = 0, 1$, $A' = 0', 1'$ we have

$$q \simeq \eta_{AA'} = \begin{bmatrix} \eta_{00'} & \eta_{01'} \\ \eta_{10'} & \eta_{11'} \end{bmatrix} = \begin{bmatrix} x_0 + \mathrm{i}x_1 & -x_2 - \mathrm{i}x_3 \\ x_2 - \mathrm{i}x_3 & x_0 - \mathrm{i}x_1 \end{bmatrix} = \begin{bmatrix} u_1 & -\bar{u}_2 \\ u_2 & \bar{u}_1 \end{bmatrix}$$

and the imaginary units of quaternions are represented by the Pauli matrices

$$\mathbf{i} \simeq \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \qquad \mathbf{j} \simeq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{k} \simeq \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad \mathrm{i} = \sqrt{-1}.$$

We define the Cauchy–Fueter operator (see [22]) as

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2} + \mathbf{k}\frac{\partial}{\partial x_3}$$

with obvious meaning of the symbols. Differentiable functions which belong to the kernel of $\partial/\partial\bar{q}$ are called regular functions. With the previous notation, the Cauchy–Fueter operator becomes

$$\frac{\partial}{\partial \bar{q}} \simeq \nabla_{AA'} = \begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix} = \begin{bmatrix} \partial_{x_0} + \mathrm{i}\partial_{x_1} & -\partial_{x_2} - \mathrm{i}\partial_{x_3} \\ \partial_{x_2} - \mathrm{i}\partial_{x_3} & \partial_{x_0} - \mathrm{i}\partial_{x_1} \end{bmatrix},$$

while the regularity condition becomes

$$\begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix} \begin{bmatrix} f_0 + \mathrm{i}f_1 & -f_2 - \mathrm{i}f_3 \\ f_2 - \mathrm{i}f_3 & f_0 - \mathrm{i}f_1 \end{bmatrix} = 0,$$

where a function $f : \mathbb{H} \to \mathbb{H}$ is written as $f = f_0 + \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$. Then, using the spinor reduction, we can write it in the form

$$\begin{bmatrix} \nabla_{00'} & \nabla_{01'} \\ \nabla_{10'} & \nabla_{11'} \end{bmatrix} \begin{bmatrix} \varphi^{0'} \\ \varphi^{1'} \end{bmatrix} = 0 \tag{1}$$

where we have set $\varphi^{0'} := f_0 + \mathrm{i}f_1$ and $\varphi^{1'} := f_2 - \mathrm{i}f_3$. In a more compact way, the two equations in (1) can be written as

$$\nabla_{AA'}\varphi^{A'} = 0, \quad A = 0, 1.$$

Remark 2.1. The indices in the function can be written up or down according to their variance or covariance. If we define the matrix

$$\epsilon_{A'B'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

we get the morphism that brings down the indices and allows the use of only covariant symbolism:

$$\varphi_{A'} = \epsilon_{A'B'}\varphi^{B'}.$$

This notation is also useful to rewrite the regularity equation $\nabla_{AA'}\varphi^{A'} = 0$ in another equivalent way as

$$\nabla_{A[A'}\varphi_{B']} = 0$$

where the symbol $[\cdot, \cdot]$ denotes the anti-symmetrization of the two Roman indices, i.e. $\nabla_{A[A'}\varphi_{B']} = \nabla_{A\,A'}\varphi_{B'} - \nabla_{A\,B'}\varphi_{A'}$. This last equation allows direct computations by making symmetrizations and anti-symmetrizations with respect to the Roman indices.

Remark 2.2. Through the paper we will use two different types of notation according to our needs. The symbol $\eta^{\ell}_{AA'}$, $A, A' = 0, 1, \ell = 1, \dots, n$ is often written as $\eta_{\alpha A'}$ with $A' = 0, 1, \alpha = 1, \dots, 2n$, α denotes the couple of indices (ℓ, A). The same notation applies to the operators $\nabla^{\ell}_{AA'}$ that become $\nabla_{\alpha A'}$. We point out that Roman capital letters always vary within the set $\{0, 1\}$, small italic letters vary between 1 and n, while Greek letters range over 1 and $2n$.

3. Algebraic approach

Another possible description of the regularity condition can be given in real components: a function f is left regular if and only if its four real components f_0, f_1, f_2, f_3 satisfy the following 4×4 system of linear constant coefficients system of differential equations given by

$$\begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0.$$

To simplify the notation, we will write the previous condition as a matrix multiplication:

$$U(D)\vec{f} = 0$$

and, when considering several quaternionic variables $q_t = x_{t0} + \mathbf{i}x_{t1} + \mathbf{j}x_{t2} + \mathbf{k}x_{t3}$, we will write $U_t(D)\vec{f} = 0$.

If we consider n Cauchy–Fueter operators we get a system of the form

$$\begin{bmatrix} U_1(D) \\ \vdots \\ U_n(D) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = P(D)\vec{f} = 0.$$

In several papers (see [1–3] and also [10]) we have studied the information contained in the module $M = \mathrm{coker}(P^t)$ where P is the $4n \times 4$ matrix symbol of $P(D)$. The matrix P has entries in the ring of polynomials $R = \mathbb{C}[x_{10}, x_{11}, x_{12}, x_{13}, \ldots, x_{n0}, x_{n1}, x_{n2}, x_{n3}]$ and it is obtained taking the Fourier transform of $P(D)$ up to a multiplication by $-\sqrt{-1}$. Strictly speaking, one would have to use dual variables, but with an abuse of notation we will always use the same variables $x_{t\ell}$.

The non-commutativity of the quaternionic setting makes non trivial the construction of the minimal free resolution of M. Note that $M = R^4/\langle P^t\rangle$ where $\langle P^t\rangle$ denotes the submodule of R^4 generated by the columns of P^t. A finite resolution of the module M can always be constructed according to what is usually called Hilbert's syzygy theorem. This fundamental result can be stated as follows.

Theorem 3.1. *There exists an integer $m \leq 4n$ and a finite exact resolution of the module M with free modules as follows*:

$$0 \longrightarrow R^{r_m} \xrightarrow{P^t_{m-1}} R^{r_{m-1}} \longrightarrow \cdots \xrightarrow{P^t_1} R^{r_1} \xrightarrow{P^t} R^{r_0} \longrightarrow M \longrightarrow 0. \quad (2)$$

The maps which appear in this resolutions are called the syzygies of M, and they can be constructed in several different ways, so that the importance of the result is not the existence of a resolution, but the fact that one can find a finite resolution, as well as the fact that we have a natural bound on its length. Note, however, that such a resolution is not unique.

One can then dualize such a resolution through the use of the Hom functor (essentially we take the duals of the spaces involved, we take the transpose of the matrices representing the operators, and we reverse the arrows) to obtain:

$$0 \longrightarrow R^{r_0} \xrightarrow{P} R^{r_1} \xrightarrow{P_1} \cdots \longrightarrow R^{r_{m-1}} \xrightarrow{P_{m-1}} R^{r_m} \longrightarrow 0. \quad (3)$$

Since the Hom functor is not exact, the complex we have just obtained is not necessarily exact, so that one can consider its cohomology by taking the quotients of kernels and images. The quotient groups are actually R-modules:

Definition 3.2. The Ext-modules of M are defined as (setting $P_0 = P$, $P_{-1} = 0$):

$$\mathrm{Ext}^j(M, R) = H^j(M, R) = \frac{\ker(P_j)}{\mathrm{im}(P_{j-1})}, \quad j = 0, 1, \ldots, m-1.$$

Remark 3.3. It is a fundamental point of the entire theory the fact that while the syzygies of Hilbert's theorem are not uniquely defined, the Ext-modules are uniquely determined by M and R, and thus are the invariant algebraic objects. They encode important analytic information. For example, the vanishing of $\mathrm{Ext}^1(M, R)$ is equivalent to the removability of compact singularities, i.e., to the Hartogs' phenomenon.

Remark 3.4. Sequence (3) is very important to analysts because it can be given an immediate analytical reading. If $\mathcal{S}$ is a sheaf of (generalized) functions, and if we maintain the same notations used up to now, we have that for every open convex set U, the following sequence is exact [14]:

$$0 \longrightarrow \mathcal{S}^P(U) \longrightarrow \mathcal{S}^{r_0}(U) \xrightarrow{P(D)} \mathcal{S}^{r_1}(U) \longrightarrow \cdots \longrightarrow \mathcal{S}^{r_{m-1}}(U) \longrightarrow \mathcal{S}^{r_m}(U) \longrightarrow 0, \tag{4}$$

where $\mathcal{S}^P$ denotes the sheaf of solutions in the sheaf $\mathcal{S}$ to the equation $P(D)f = 0$.

The general result in the case of the Cauchy–Fueter complex (see [10]) is contained in the following theorem summarizing the results contained in [2,3]:

Theorem 3.5. *Let M be the module associated to the Cauchy–Fueter system in $n > 1$ variables. Then its resolution is*

$$0 \longrightarrow R^{r_{2n-1}}(-2n) \xrightarrow{P^t_{2n-2}} R^{r_{2n-2}}(-2n+1) \longrightarrow \cdots \longrightarrow R^{r_3}(-4)$$
$$\longrightarrow R^{r_2}(-3) \xrightarrow{P^t_1} R^{4n}(-1) \xrightarrow{P^t} R^4 \longrightarrow M \longrightarrow 0.$$

In particular:

I) *the resolution of M has length $2n-1$,*
II) *all the maps in the resolutions are linear except the first one.*
Moreover,
III) *the Betti number r_ℓ at the step ℓ is given by*

$$r_\ell = 4\binom{2n-1}{\ell}\frac{n(\ell-1)}{\ell+1}.$$

Sketch of the proof. The proof of the theorem is exquisitely algebraic and it is contained in [2] and [3]. First of all one can compute the reduced Gröbner basis of the R-module $\langle P^t\rangle$ that contains the columns of P^t and the columns of the matrices $[U_t, U_s]$; then one observe that the variables x_{11}, x_{n2}, x_{i3} form an M-regular sequence. Next, one can define the module:

$$M^* = \frac{R^4}{\langle U_t, U_rU_s - U_sU_r, x_{11}\mathbf{e}_\ell, x_{n2}\mathbf{e}_\ell, x_{i3}\mathbf{e}_\ell\rangle_{i=1,\dots,n}},$$
$$1 \le r < s \le n,\ \ \ell = 1, 2, 3, 4$$

where $\mathbf{e}_i$ denotes the four components vector having zero entries except the ith that equals 1. Let $\wp$ be the maximal ideal of R generated by all the $4n$ variables. Then, the polynomials $x_{21}+x_{12}+\cdots+x_{i1}+x_{i-1,2}+\cdots+x_{n1}+x_{n-1,2}$ form a maximal M^* regular sequence in $\wp$. Now using the Auslander–Buchsbaum formula one obtains that the module M has projective dimension equal to $2n-1$ so the length of a minimal free resolution of M is $2n-1$.

The fact that the first syzygies are quadratic follows from their explicit description (see [2]), while the proof of the case of higher order syzygies follows from the computation of the Castelnuovo–Mumford regularity of M.

Finally the Betti numbers r_0 and r_1 are trivially equal to 4 and $4n$, while r_i for $2 \le i \le 2n-1$ can be computed by equating the coefficients of the Hilbert–Poincaré series written using the minimal free resolution of M and the fact that the Hilbert–Poincaré series of M is given by (see [2]):

$$\mathcal{P}_n(t) = \frac{4+4(n-1)t}{(1-t)^{2n+1}}. \qquad \square$$

Remark 3.6. The dual of the resolution arising from the Hilbert syzygy theorem is a complex that, in general, is not exact. In this particular case the complex is

$$0 \longrightarrow R^4 \xrightarrow{P} R^{4n} \xrightarrow{P_1} R^{r_2} \longrightarrow \cdots \longrightarrow R^{r_{2n-2}} \xrightarrow{P_{2n-2}} R^{r_{2n-1}} \longrightarrow 0. \tag{5}$$

By a well known result (see [16] Corollary 1, p. 337) we have immediately the following:

Theorem 3.7. *The complex* (5) *is exact except at the last spot, i.e.* $\mathrm{Ext}^j(M,R)=0$ *for* $j=1,\ldots,2n-2$, $\mathrm{Ext}^{2n-1}(M,R)\neq 0$.

As was explained in Remark 3.4, Theorem 3.5 implies that on the side of analysis, we get the following analogue of the Dolbeault complex (see [10]).

Theorem 3.8. *Let U be a convex open (or convex compact) set in $\mathbb{R}^{4n}$ and let $\mathcal{S}$ be the sheaf of infinitely differentiable functions. The complex described in* Theorem 3.5 *induces the exact complex*:

$$0 \longrightarrow \mathcal{S}^P(U) \longrightarrow \mathcal{S}^4(U) \xrightarrow{P(D)} \mathcal{S}^{4n}(U) \longrightarrow \cdots \longrightarrow \mathcal{S}^{r_{2n-2}}(U) \longrightarrow \mathcal{S}^{r_{2n-1}}(U) \longrightarrow 0, \tag{6}$$

starting with the operator $P(D)$, which is given by the Cauchy–Fueter operators in all the n variables.

4. Parabolic geometries

The notion of parabolic geometry was introduced in [13], following the Fefferman concept of parabolic invariant theory in [11,12]. These geometries in their flat version go back to Klein's definition of geometry as the study of homogeneous spaces G/P. Parabolic geometries are modelled on a homogeneous space $M = G/P$, where G is a semisimple Lie group and P its parabolic subgroup.

Cartan then created a curved version of such geometry on a manifold M. In this curved version, he replaced the principal fiber bundle $p : G \to G/P$ of the homogeneous model with a general principal fiber bundle $\mathcal{G} \to M$ with the fiber group P and he replaced the Maurer–Cartan connection ω (which is a one-form on G with values in $\mathfrak{g}$, its Lie algebra) by a one-form ω on $\mathcal{G}$ with values in $\mathfrak{g}$, having suitable properties deduced from properties of the Maurer–Cartan form. This form ω on $\mathcal{G}$ is called the Cartan connection, and in a sense it plays the role that the Levi–Civita connections play in Riemannian geometry. The couple $(\mathcal{G}, \omega)$ is then called a parabolic structure on the manifold M, modeled on the homogeneous space G/P. The chosen manifold M can have several different parabolic structures. For example, the sphere can be considered as a manifold with a given projective, conformal, or quaternionic structure. A nice introduction to parabolic geometries can be found in [18].

For our purposes, it is sufficient to consider homogeneous models $M = G/P$ with the corresponding Maurer–Cartan form. Even more, we shall work only on a big cell inside M. A big cell in M is a vector space V embedded in M, which is an open and dense subspace of M. In conformal geometry (which is the most typical example of a parabolic geometry), M is a sphere S^n of dimension n and V is $\mathbb{R}^n$ embedded into S^n by stereographic projection.

We shall consider here another example of a parabolic geometry, the so called quaternionic geometry. Its homogeneous model is the quaternionic projective n-space $\mathbb{P}^n(\mathbb{H})$ and the big cell inside is the space $\mathbb{H}^n$ of n quaternionic variables embedded into $\mathbb{P}^n(\mathbb{H})$.

For each linear representation $\mathbb{E}$ of the (parabolic) structure group P, there is the associated homogeneous vector bundle $E(G/P)$ over the corresponding homogeneous space G/P. The bundle $E(G/P)$ is defined as the quotient $G \times \mathbb{E}/\sim$, where the equivalence relation is defined by

$$(g, e) \sim (gp, p^{-1}e), \quad g \in G, \ e \in \mathbb{E}, \ p \in P.$$

There is a natural action of the group G on the vector bundle $E(G/P)$, induced by the left action $g' \cdot (g, e) = (g'g, e)$, $g, g' \in G$, $e \in \mathbb{E}$. Hence there is also the induced action of G on sections of bundles $E(G/P)$. Invariant operators on $M = G/P$ are then defined as those operators on such sections, which commute with the above actions. We shall consider only invariant differential operators. When working on the big cell, vector bundles in question are trivial bundles $V \times \mathbb{E}$.

An important fact to note is that such invariant differential operators are rare beings and that their classification is known in many cases, if operators acts between bundles associated to irreducible representations of P. The classification was found using tools of representation theory (Verma modules and their morphisms, see [6,7]). Irreducible representations of P coincide with irreducible representations of the Levi factor G_0 of P. The Levi factor G_0 is a reductive group and its irreducible representations are well understood.

Consider differential operators from $\mathcal{C}^\infty(V, \mathbb{E}_1)$ to $\mathcal{C}^\infty(V, \mathbb{E}_2)$. An invariant differential is characterized by a choice of its source and target (i.e., by a choice of irreducible representations $\mathbb{E}_1$ and $\mathbb{E}_2$) up to a constant multiple. But for most of choices of $\mathbb{E}_1$ and $\mathbb{E}_2$, there is no invariant differential operator at all! Irreducible representations of G_0 are classified by their highest weights. A very useful fact is that such invariant differential operators can act only between points of an orbit of a finite group on the space of weights. Hence our choice of values for invariant operators is also enormously constrained. We shall describe these facts in more details in the case of quaternionic geometry. Similar facts for other parabolic geometries can be found in [5].

We shall use here only invariant differential operators of order one and two. A general description of first order invariant differential operators on any manifold with a given parabolic geometry can be found in [19]. Similar questions are much more complicated for higher order operators. A description of a large class of such invariant operators can be found in [9]. We shall need here only one simple case of a second order operator. Let us now describe the construction of all invariant first order operators and some invariant second order operators.

4.1. Invariant first order operators

We want to describe explicitly the form of first order invariant systems of differential operators for flat models of parabolic geometries (see [19]). Suppose that G is a real semisimple Lie group and P its parabolic subgroup. Let $M = G/P$ be the corresponding homogeneous space. We shall treat only the $|1|$-graded case (for more general cases, see [19]).

We shall suppose that the Lie algebra $\mathfrak{g}$ of G has a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the Lie algebra of the Levi factor G_0 of P. The sum $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a parabolic subalgebra of $\mathfrak{g}$, and it is the Lie algebra of P. The subalgebra $\mathfrak{g}_{-1}$ can be considered as a representation of P through the identification $\mathfrak{g}_{-1} \equiv \mathfrak{g}/\mathfrak{p}$. The vector space $V = \mathfrak{g}_{-1}$ can be embedded (using the exponential map) into M as a big cell. Moreover, the tangent space TM is associated to the vector bundle corresponding to the module $\mathfrak{g}_{-1}$. Similarly, the P-module $\mathfrak{g}_1$ is the model for the cotangent bundle, which means that the cotangent bundle T^*M is associated to the P-module $\mathfrak{g}_1$. Both P-modules $\mathfrak{g}_{\pm 1}$ are irreducible, the nilpotent part is acting trivially.

Let $\mathbb{E}$, $\mathbb{F}$ be two irreducible G_0-modules. The group G_0 is reductive and can be written as a product of a semisimple part G_0^s and a commutative group $\mathbb{R}_+$. Hence every irreducible representation of G_0 is the tensor product of a one-dimensional representations of $\mathbb{R}_+$ (specified by a real number w, it is a generalization of a conformal weight from the case of conformal geometry) and an irreducible representation of G_0^s, specified by its highest weight.

Suppose that $\mathbb{E}$ is an irreducible G_0^s-module and denoted by $\mathbb{C}_w$, $w \in \mathbb{R}$, the one-dimensional representation of $\mathbb{R}^+$ on $\mathbb{C}$, given by multiplication with the factor λ^w, $\lambda \in \mathbb{R}^+$. Then $\mathbb{E}(w)$ will denote an irreducible G_0 (hence P)-module $\mathbb{E} \otimes \mathbb{C}_w$. Any irreducible P-module can be written in such a way. A comfortable way to encode both pieces of information for such P-modules is to use the weight λ in the dual of the Cartan subalgebra of the whole Lie algebra $\mathfrak{g}$. We will describe this below in more details for the case of quaternionic geometry.

Let us now consider first order invariant differential operators between smooth maps defined on domains in $V = \mathfrak{g}_{-1}$ with values in $\mathbb{E}$, resp. $\mathbb{F}$, where both modules are G_0^s-modules, the weights w will be specified later. Suppose that $\mathbb{E}$ is given. Then there is only a finite number of possibilities for invariant first order differential operators acting on such values. They are all constructed by the following procedure.

Take $f \in \mathcal{C}^\infty(V, \mathbb{E})$. Then its gradient ∇f belongs to $\mathcal{C}^\infty(V, \mathfrak{g}_1 \otimes \mathbb{E}_1)$. It is possible to show that the product $\mathfrak{g}_1 \otimes \mathbb{E}_1$ of G_0^s-modules decomposes in a unique way into irreducible components:

$$\mathfrak{g}_1 \otimes \mathbb{E} = \mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_k,$$

where there are no multiplicities in the decomposition. Denote by π_i the projections of $\mathfrak{g}_1 \otimes \mathbb{E}$ to $\mathbb{F}_i$, $i = 1, \ldots, k$.

Then for every $i = 1, \ldots, k$ there exists a unique number $w_i \in \mathbb{R}$ such that $D_i = \pi_i \circ d$ is an invariant first order differential operator, mapping smooth maps with values in $\mathbb{E}(w_i)$ to smooth maps with values in $\mathbb{F}_i(w_i - 1)$. Any other invariant first order operator on sections of $\mathbb{E}$ is isomorphic to an operator of the type D_i. So we see that to find all first order operators, it is necessary only to be able to decompose the tensor product of two irreducible G_0^s-modules into irreducible components (one-dimensional representations of the commutative part play no role in the decomposition). There are standard techniques available for such decompositions and the result is known in all cases.

4.2. Certain invariant second order operators

A description of invariant second order operators is a much more complicated question. There are certain constructions available for higher order invariant operators (see e.g. [9]) but they do not cover the case we need.

On the other hand, we can often find suitable candidates by following a procedure similar to the one used above. We shall describe it now. Suppose we want to construct an invariant second order operator on functions on V with values in an irreducible G_0^s-module $\mathbb{E}$. Let us consider again the splitting:

$$\mathfrak{g}_1 \otimes \mathbb{E} = \mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_k$$

and similarly

$$\mathfrak{g}_1 \otimes \mathbb{F}_i = \mathbb{F}_{i1} \oplus \cdots \oplus \mathbb{F}_{il_i}.$$

Then

$$\mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathbb{E} = \oplus_{ij} \mathbb{F}_{ij}$$

is a decomposition of the left hand side into irreducible pieces. This time, however, there can be higher multiplicities, i.e., certain summands can be isomorphic as G_0-modules. Some of the summands will appear with a multiplicity one. If $\mathbb{F}_{ij}$ is such a summand, and π_{ij} is the corresponding invariant projection, the operator $f \to \pi_{ij}(\nabla\nabla f)$ is the only possible candidate for an invariant second order operator from sections of E to sections of F_{ij}. The remaining question is whether a number w in the definition of $\mathbb{E}$ can be chosen in such a way that the operator is invariant. We shall not discuss this question in detail.

5. Orbits of the Weyl group in the weight spaces

As explained above, irreducible representations of G_0 can be characterized by a weight λ in the dual $\mathfrak{h}^*$ of the Cartan subalgebra of $\mathfrak{g}$. A general statement (proved by representation theory) says that an invariant operator from $\mathcal{C}^\infty(V, \mathbb{E}_\lambda)$ to $\mathcal{C}^\infty(V, \mathbb{E}_\mu)$ can exist only if both weights are on the same orbit of the affine Weyl action on $\mathfrak{h}^*$. Due to the fact that W is a finite group, it gives just a finite number of candidates for a given $\mathbb{E}_\lambda$. We shall describe it in more details below in the case of interest.

5.1. The case of quaternionic geometry

Our setting for invariant quaternionic complexes is the quaternionic geometry. The group G is the projective group of quaternionic projective geometry.

Consider the projective space $\mathbb{P}_n(\mathbb{H})$. This is the quotient of the space $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation:

$$(q_0, \dots, q_n) \equiv (q_0 r, \dots, q_n r),\ r \in \mathbb{H} \setminus \{0\}.$$

The group $GL(n+1, \mathbb{H})$ of invertible quaternionic matrices is acting on $\mathbb{P}_n(\mathbb{H})$ in an obvious way. The action has a kernel and the quotient of $GL(n+1, \mathbb{H})$ by the kernel is the group of projective transformations G. Its Lie algebra is the vector space of $(n+1) \times (n+1)$ quaternionic matrices with 0 trace.

The grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is given by a block decomposition of $\mathfrak{g}$ induced by the decomposition of $\mathbb{H}^{n+1} = \mathbb{H} \oplus \mathbb{H}^n$. The diagonal part is the Lie algebra $\mathfrak{g}_0$, while strictly lower triangular matrices form a commutative algebra $\mathfrak{g}_{-1}$ and strictly upper triangular matrices form a commutative algebra $\mathfrak{g}_1$. The commutativity of those matrices shows immediately that the decomposition is indeed a grading.

The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can be chosen to be the diagonal subalgebra:

$$H \in \mathfrak{g} | H = \mathrm{diag}(q_0 \dots, q_n), \qquad \sum_0^n q_i = 0, q_i \in \mathbb{C} \subset \mathbb{H}$$

where $\mathbb{C} \subset \mathbb{H}$ is given by quaternions of the form $q = x_0 + \mathbf{i} x_1$. It is clearly a maximal commutative subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}_0$.

All representations we shall consider will be complex representations. Hence they will be at the same time representations of the complexification $\mathfrak{g}_0^{\mathbb{C}}$. This Lie algebra $\mathfrak{g}_0$ is a sum $\mathfrak{g}_0 = \mathrm{sl}(1, \mathbb{H}) \oplus \mathrm{sl}(n, \mathbb{H}) \oplus \mathbb{R}$ and its complexification $\mathfrak{g}_0^{\mathbb{C}}$ is a reductive complex Lie algebra equal to the sum $\mathrm{sl}(2, \mathbb{C}) \oplus \mathrm{sl}(2n, \mathbb{C}) \oplus \mathbb{C}$.

The complexification $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra in $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$. Any irreducible representation of $\mathfrak{g}_0^{\mathbb{C}}$ can be written as a tensor product of an irreducible representation of $\mathrm{sl}(2, \mathbb{C})$ with an irreducible representation of $\mathrm{sl}(2n, \mathbb{C})$ and with a one-dimensional representation of the commutative part $\mathbb{C}$.

Suppose that $\omega_i \in (\mathfrak{h}^{\mathbb{C}})^*$, $i = 1, \dots, 2n-1$, are fundamental weights for the Lie algebra $\mathrm{sl}(2n, \mathbb{C})$. Then any irreducible representation of the reductive Lie algebra $\mathfrak{g}_0^{\mathbb{C}}$

can be uniquely characterized by its highest weight $\lambda = \sum_{i=1}^{2n-1} \lambda_i \omega_i$ with $\lambda_i \in \mathbb{Z}, \lambda_i \geq 0$, $i = 1, 3, 4, \ldots, 2n-1$ and $\lambda_2 \in \mathbb{C}$. Hence we shall use the sequence of coefficients $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2n-1})$ as the label of the corresponding irreducible $\mathfrak{g}_0$-module.

Here are some important examples. The defining representation $\mathbb{E} \simeq \mathbb{C}^2$ of $\mathrm{sl}(2, \mathbb{C})$ corresponds to the highest weight $(1, 0, 0, \ldots, 0)$. The weight $(j, 0, \ldots, 0)$ corresponds to the symmetric power $\odot^j(\mathbb{C}^2)$ of the previous representation. The weight $(0, 1, 0, \ldots, 0)$ corresponds to a one-dimensional representation of the commutative factor of $\mathfrak{g}_0$. We shall denote it by $\mathbb{C}[1]$, its dual by $\mathbb{C}[-1]$ and their powers by $\mathbb{C}[k]$, $k \in \mathbb{Z}$. A tensor product of a representation $\mathbb{V}$ with $\mathbb{C}[-k]$ will be denoted by $\mathbb{V}[-k]$. The weights $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the $(k+2)$th place correspond to the exterior power $\Lambda^k(\mathbb{C}^{2n})$ of the defining representation of $\mathbb{C}^{2n}$ of $\mathrm{sl}(2n, \mathbb{C})$.

Let W be the Weyl group of the Lie algebra $\mathrm{sl}(2n+2, \mathbb{C})$. This is a finite group generated by reflections in $(\mathfrak{h}^{\mathbb{C}})^*$. Let $\delta = \sum_{i=1}^{2n-1} \omega_i$ be the sum of all fundamental weights. The affine action of an element w on $(\mathfrak{h}^{\mathbb{C}})^*$ is defined as

$$w \cdot \lambda := w(\lambda + \delta) - \delta.$$

The key information on invariant operators tells us that if there is an invariant operator (for a given parabolic geometry) from maps with values in $\mathbb{E}_\lambda$ to maps with values in $\mathbb{E}_\mu$, than both highest weights $\lambda, \mu \in (\mathfrak{h}^{\mathbb{C}})^*$ should belong to the same orbit of the affine action of the Weyl group. Hence we have always a finite number of such weights.

An orbit of the Weyl group W is called regular if the action of W on the orbit is free. The orbit is called singular if some of its points are fixed by a nontrivial element of the Weyl group (in another words, if some elements of the orbit belong to walls of fundamental domains of the Weyl group). Affine orbits can be divided into regular and singular types. The regular ones are those for which the elements of the form $\lambda + \delta$ form a regular orbit. In the opposite case, the affine orbit is said to be of a singular type.

5.2. *The case of n Fueter operators*

Denote by $\partial_{\bar{q}}$ the Fueter operator for quaternionic valued functions in one variable q. Consider now the operator D_0 on the space of quaternionic functions f of n quaternionic variables $(q_1, \ldots, q_n) \in \mathbb{H}^n$ given by $D_0 f = (\partial_{\bar{q}_1} f, \ldots, \partial_{\bar{q}_n} f)$.

To compare it with invariant operators on the projective quaternionic space, we shall identify the quaternionic representation $\mathbb{H}$ of $\mathrm{sp}(1, \mathbb{H})$ with the complex representation $\mathbb{C}^2$ of the complexification $\mathrm{sl}(2, \mathbb{C})$. Similarly, we shall identify the quaternionic representation $\mathbb{H}^n$ of $\mathrm{sl}(n, \mathbb{H})$ with the complex representation $\mathbb{C}^{2n}$ of the complexification $\mathrm{sl}(2n, \mathbb{C})$. The first representation corresponds to the highest weight $(1, w, 0, \ldots, 0)$; the second one to the highest weight $(0, w', 1, 0, \ldots, 0)$. There can be an invariant operator between maps into such spaces only if these weights are at the same affine orbit of the Weyl group. It fixes weights (and corresponding representations) to $\lambda_0 := (1, -2, 0, \ldots, 0) \simeq \mathbb{C}^2[-2]$, resp. $\lambda_1 := (0, -3, 1, 0, \ldots, 0) \simeq \mathbb{C}^{2n}[-3]$. We know that the operator D should be of the first order. It is possible to check directly that the construction of first order invariant operators described above defines indeed the operator D_0.

In more details, if f has values in the module $\mathbb{E}_{\lambda_1}$, then its differential has values in $\mathbb{E}_{\lambda_0} \otimes \mathbb{E}_\mu$, with $\mu = (1, -2, 1, 0 \ldots, 0)$. The tensor product decomposes as

$$\mathbb{E}_{\lambda_0} \otimes \mathbb{E}_\mu = \mathbb{E}_{\lambda_{11}} \oplus \mathbb{E}_{\lambda_{12}},$$

with $\lambda_{11} = (2, -4, 1, 0, \ldots, 0)$ and $\lambda_{12} = (0, -3, 1, 0, \ldots, 0)$. The projection onto the second summand leads to the invariant operator D_0.

Now, it is possible to write down the whole affine orbit starting with the weight λ_0. The list of points on the orbit is

$$\lambda_2 = (0, -4, 0, 0, 1, 0, \ldots, 0), \qquad \lambda_3 = (1, -5, 0, 0, 0, 1, 0, \ldots, 0).$$

The general member of the list is

$$\lambda_j = (j-2, -j-2, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with the 1 on the $(j+3)$th place for $j = 2, \ldots, 2n-1$. The last two weights of the orbit are hence

$$\lambda_{2n-2} = (2n-4, -2n, 0, \ldots, 0, 1), \qquad \lambda_{2n-1} = (2n-3, -2n+1, 0, \ldots, 0).$$

Hence the corresponding modules are

$$\begin{aligned} \mathbb{E}_{\lambda_2} &\simeq \Lambda^3(\mathbb{C}^{2n})[-4], \\ \mathbb{E}_{\lambda_3} &\simeq \mathbb{C}^2 \otimes \Lambda^4(\mathbb{C}^{2n})[-5]; \cdots; \\ \mathbb{E}_{\lambda_j} &\simeq \odot^{j-2}(\mathbb{C}^2) \otimes \Lambda^{j+1}(\mathbb{C}^{2n})[-j-2]; \cdots; \\ \mathbb{E}_{\lambda_{2n-1}} &\simeq \odot^{2n-3}(\mathbb{C}^2) \otimes \Lambda^{2n}(\mathbb{C}^{2n})[-2n-1]. \end{aligned}$$

5.3. The construction of the differential operators in the sequence

Now we shall describe the operators D_j, $j = 0, 1, \ldots, 2n-4$ in the resulting sequence. Elements of the representation $\mathbb{C}^2$ will be denoted by $\varphi^{A'}$, $A' = 0, 1$. Elements of the symmetric power $\odot^j(\mathbb{C}^2)$ are symmetric tensor field $\varphi^{A' \ldots E'}$ with j capital roman indices. Elements of the outer power $\Lambda^k(\mathbb{C}^{2n})$, $k = 1, \ldots, 2n-1$ are antisymmetric tensor fields $\varphi_{\alpha,\ldots,\gamma}$ with k Greek indices. In Section 2, we have already introduced the symbol $\nabla_{A'\alpha}$, $A' = 0, 1, \alpha = 1, \ldots, 2n$ for the gradient.

The operator D_0 from functions with values in $\mathbb{E}_{\lambda_0}$ to functions with values in $\mathbb{E}_{\lambda_1}$ can be written as

$$[D_0(\varphi^{A'})]_\alpha = \nabla_{A'\alpha}\varphi^{A'}.$$

The operator D_1 is defined by

$$[D_1(\varphi_\gamma)]_{\alpha\beta\gamma} = \nabla_{A'[\alpha}\nabla_\beta^{A'}\varphi_{\gamma]}, \tag{7}$$

where the brackets [...] mean anti-symmetrization in the corresponding indices. Note that this is a second order operator.

All other operators are of first order. The operator D_j is defined on fields with $j-2$ upper indices and $j+1$ lower indices by

$$[D_j(\varphi^{B'\ldots F'}_{\beta\ldots\delta})]^{A'\ldots F'}_{\alpha\ldots\delta} = \nabla^{(A'}_{[\alpha}\varphi^{B'\ldots F')}_{\beta\ldots\delta]}, \tag{8}$$

where the round parentheses (...) mean the symmetrization in the corresponding indices. We can summarize the contents of this section into the following theorem:

Theorem 5.1. *There is an exact complex of invariant differential operators D_i, $i = 0, \ldots, 2n-2$ acting from the spaces $\mathbb{E}_{\lambda_i}$ to $\mathbb{E}_{\lambda_{i+1}}$ where $\mathbb{E}_{\lambda_0} = \mathbb{C}^2$, $\mathbb{E}_{\lambda_1} = \mathbb{C}^{2n}$ and $\mathbb{E}_{\lambda_j} \simeq \odot^{j-2}(\mathbb{C}^2) \otimes \Lambda^{j+1}(\mathbb{C}^{2n})$ and where D_0 is associated to $n \geq 2$ Cauchy–Fueter operators. All the operators are of the first order, except D_1 which is described in (7). All the other first order operators for $j \geq 2$ are obtained as in (8). The associated exact complex is then*

$$\begin{aligned} 0 &\longrightarrow \mathbb{C}^2 \xrightarrow{D_0} \mathbb{C}^{2n} \xrightarrow{D_1} \Lambda^3(\mathbb{C}^{2n}) \xrightarrow{D_2} \mathbb{C}^2 \otimes \Lambda^4(\mathbb{C}^{2n}) \longrightarrow \cdots \\ &\longrightarrow \odot^{2n-3}(\mathbb{C}^2) \otimes \Lambda^{2n}(\mathbb{C}^{2n}) \longrightarrow 0. \end{aligned} \tag{9}$$

This sequence will be rewritten to define a complex of maps at the algebraic level in the next section. For an explicit description in the case of two and three operators (this latter case will recover the general procedure for n operators), see the paper [8].

6. Invariant resolution for several Fueter operators

In the language of invariant operator theory, we can describe the Cauchy–Fueter complex starting with the $2n \times 2$ matrix associated to n Cauchy–Fueter operators:

$$\begin{bmatrix} V_1(D) \\ \vdots \\ V_n(D) \end{bmatrix} \epsilon_{A'B'} \begin{bmatrix} \varphi^{0'} \\ \varphi^{1'} \end{bmatrix} = Q(D)\vec{\varphi} = 0,$$

where

$$V_t(D) = \begin{bmatrix} \partial_{x_{t0}} + i\partial_{x_{t1}} & -\partial_{x_{t2}} - i\partial_{x_{t3}} \\ \partial_{x_{t2}} - i\partial_{x_{t3}} & \partial_{x_{t0}} - i\partial_{x_{t1}} \end{bmatrix}$$

and

$$Q(D) = \begin{bmatrix} \vdots & \vdots \\ -\partial_{x_{t2}} - i\partial_{x_{t3}} & -\partial_{x_{t0}} - i\partial_{x_{t1}} \\ \partial_{x_{t0}} - i\partial_{x_{t1}} & -\partial_{x_{t2}} + i\partial_{x_{t3}} \\ \vdots & \vdots \end{bmatrix}.$$

At the level of symbols we get the matrix Q, with entries in R, that we can write in a more compact way as

$$Q = \begin{bmatrix} \vdots & \vdots \\ \eta^t_{01} & -\eta^t_{00} \\ \eta^t_{11} & -\eta^t_{10} \\ \vdots & \vdots \end{bmatrix}.$$

The resolution we get is the analogue to the one described in Theorem 3.5: in this language the only difference is that all the maps need to be translated into complex relations and all the Betti numbers are divided by two, i.e., $r'_0 = 2$, $r'_1 = 2n$ and

$$r'_\ell = 2\binom{2n-1}{\ell}\frac{n(\ell-1)}{\ell+1}.$$

In particular, Theorem 3.8 becomes:

Theorem 6.1. *Let U be a convex open (or convex compact) set in $\mathbb{R}^{2n} = \mathbb{C}^n$ and let $\mathcal{S}$ be the sheaf of infinitely differentiable functions. The complex described in Theorem 3.5 induces the exact complex*

$$\mathcal{S}^2(U) \xrightarrow{Q(D)} \mathcal{S}^{2n}(U) \longrightarrow \cdots \longrightarrow \mathcal{S}^{r'_{2n-2}}(U) \longrightarrow \mathcal{S}^{r'_{2n-1}}(U) \longrightarrow 0, \qquad (10)$$

starting with the operator $Q(D)$, which is given by the Cauchy–Fueter operators in all the n variables.

We now show that the complexes obtained according to the invariant operator theory and the one computed using algebraic tools are the same, thus giving a positive answer to the questions posed in [20] and [21] about the comparison between the two approaches. Let us begin by recalling the following well known result (see for example [15]):

Proposition 6.2. *Any two minimal graded resolutions*

$$\cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \to 0$$

and

$$\cdots \longrightarrow G_1 \xrightarrow{\varphi_1} G_0 \longrightarrow M \to 0$$

of M are isomorphic as complexes, i.e. there are graded isomorphisms $\alpha_j : F_j \longrightarrow G_j$ such that $\alpha_{j-1}\phi_j = \varphi_j\alpha_j$, for all $j \geq 1$.

Theorem 6.3. *The complexes (9) and (10) are isomorphic.*

Proof. Let us consider the complex $\mathcal{C}$ which is the Fourier transform of the dual of the complex (9). It is a complex in which the first map coincides with the map Q^t, where Q is the Fourier transform of $Q(D)$. The minimal free resolution of Q^t has the same length, Betti numbers and degrees of the maps as $\mathcal{C}$ (see Theorem 3.1 in [20]). Now, the fact that the matrices associated to the maps appearing in $\mathcal{C}$ have homogeneous entries of degree two at the first step and one in the next steps assures that the relations they represent are not redundant. The sufficiency of these relations is guaranteed by the fact that their number equals the number of relations found in the minimal free resolution. So $\mathcal{C}$ is not only a complex but, by Proposition 6.2, it is a minimal free resolution of Q^t. By duality, this proves the statement. □

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